¿Cross-Connections?

Metric Embeddings,
Graph Laplacians,
Rank Minimization,

Christoph Helmberg (TU Chemnitz)

Overview

Minimum Distortion Embeddings of Finite Metric Spaces

Further Embedding Problems

Generalized Laplacians and Nodal Domains

Low Rank Approaches

Distance, Semimetric, Metric [DL97,LLR95]

Given a (finite) set N, a distance $d : N \times N \to \mathbb{R}_+$ satisfies

- symmetry: $d(i,j) = d(j,i)$ for $i, j \in N$
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Depending on d, the pair (N,d) forms a distance/semimetric/metric space.

Example: Given a connected graph $G = (N,E)$, let $d(i,j)$ denote the length of a shortest path between i and j, then (N,d) is a metric space.

Further examples:

- MET_n in DL97: Symmetric nonnegative matrices of order n with diagonal 0 that satisfy all $\triangle$-inequs, describe all semimetrics on n elements.
  Given $d, d' \in \mathrm{MET}_n$ and $\lambda \geq 0$, we have $\lambda(d + d') \in \mathrm{MET}_n$, so MET_n is a convex cone.

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- CUT_n in DL97: For $S \subseteq N$ define the cut-semimetric

$$d_S(i,j) = \begin{cases} 1 & \text{for } ij \in \delta(S) := \{ij : |S \cap \{i,j\}| = 1\}, \\ 0 & \text{otherwise}, \end{cases}$$

  then $\sum_{S \subseteq N} \lambda_S d_S$ with $\lambda_S \geq 0$ is again a semimetric.
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- For a normed space $(E, \|\cdot\|)$, $d_{\|\cdot\|}(x,y) = \|x - y\|$ is the associated norm metric. In particular, we use $d_{l_p} = \|x - y\|_p$

Embeddings and Distortion

Given two distance spaces (N, d), (N', d') a mapping $\phi : N \longrightarrow N'$ is called a Lipschitz embedding of (N, d) into (N', d') with distortion $c \geq 1$ if

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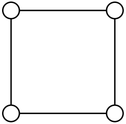
- d-cube: $n = 2^d$ vertices, distortion $\sqrt{d} \in O(\log n)$ for l_2.



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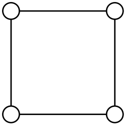

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[JohnsonLindenstrauss1984]: Any n points in an Euclidean space can be mapped to $\mathbb{R}^t$ with $t = O(\frac{\log n}{\varepsilon^2})$ and distortion $\leq 1 + \varepsilon$.

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"$\Rightarrow$": setting $v_i = (d(1,i), \ldots, d(n-1,i)) \in \mathbb{R}^{n-1}$ for $i \in N$ defines an isometry:

$$\|v_i - v_j\|_\infty = \max\{\underbrace{|d(k,i) - d(k,j)|}_{\triangle\text{-ineq } \le d(i,j)} : k \in \{1, \ldots, n-1\}\} \overset{k \in \{i,j\}}{=} d(i,j)$$

Example: l_1 Embeddability and the Cut Cone

A cut semimetric d_S, $S \subseteq N$ is l_1 embeddable:
$\phi(i) = 1$ for $i \in S$, $\phi(i) = 0$ for $i \in N \setminus S$.

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Given $d = \sum_{k=1}^{m} \lambda_k d_{S_k}$ with $\lambda_k \geq 0$, define n vectors $v^i \in \mathbb{R}^m$, $i \in N$,

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Checking l_1 embeddability is *NP*-complete!

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- An Euclidean embedding $\rho : N \to \mathbb{R}^d$ of (N, d) has distortion c if

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- Put $v_i = \rho(i)$, $V = [v_1, \ldots, v_n]$ and consider the

Gram matrix $X = V^T V \succeq 0 \quad (x_{ij} = v_i^T v_j)$ $\qquad E_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} i \\ j \end{matrix}$

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$$\begin{array}{ll} \max & t \\ \text{s.t.} & t \cdot d(i,j)^2 \le \langle E_{ij}, X \rangle \le d(i,j)^2 \qquad \forall i,j \in N, i \ne j \\ & X \succeq 0 \end{array}$$

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  Use [JL84] $\to$ embedding in $\mathbb{R}^{O(\log n)}$ with distortion $O(D_{\text{opt}})$

- [Vallentin2007] optimal distortion embeddings for classes of distance regular graphs (Hamming/Johnson Graphs/. . .)

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Sensor Network Localization Euclidean Distance Matrix Completion

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- various relaxation variants exist [SoYe2004,DKQW2006], e.g.,

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- Questions of uniqueness lead to rigidity/tensegrity theory. [SoYe2005]
- main difficulty: $\mathbb{R}^d \;\Leftrightarrow\; \operatorname{rank}(X^*) \le d$ (nonconvex!)

$\rightarrow$ general interest in "existence of low rank optimal solutions"

Graph Realizations [BelkConnelly07]

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  (sequ. of deletions of edges or isolated vertices or contractions of edges)
  - 1-realizable: no K_3 minor (forests)
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- [SoYe06] give a polynomial algorithm for constructing an (approx.) realization for 3-realizable graphs

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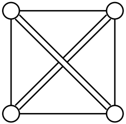

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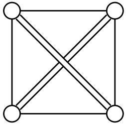

- Prestress stability: for potential functions f_{ij} for $ij \in E$ find a local min

$$\min \quad \sum_{ij \in E} f_{ij}(\|v_i - v_j\|^2) \quad \text{s.t.} \quad \|v_i - v_j\|^2 = / \leq / \geq \ldots$$

A non-deg. local min, unique up to rigid congruences, is *prestress stable*.

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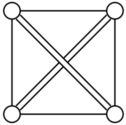

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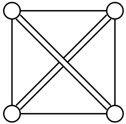

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Maximum Variance Unfolding [WeinbergerSaul2004]

- visualization-technique for graphs $G = (N, E)$ in data mining
- Spread out a graph as much as possible, edges = cables of length 1

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- Corresponds to finding a fastest mixing (symmetric) Markov chain on G [SBXD04]

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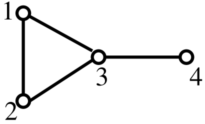
Low Rank Approaches

Laplace matrix of a graph

- simple undirected Graph $G = (N, E)$, $E \subseteq \{ij : i, j \in N, i \neq j\}$
- Laplacian $L(G) = \mathrm{Diag}(A\mathbf{1}) - A$, A ... adjacency matrix, $\mathbf{1} = [1, 1, \ldots, 1]^T$



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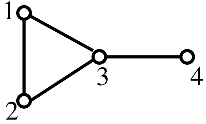

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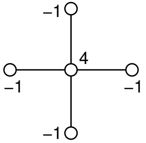
May be viewed as a discrete version of the Laplace operator $\Delta u = u_{xx} + u_{yy}$:



For the wave equation $\Delta u = u_{tt}$ solve $\Delta u + \lambda u = 0 \quad \leftrightarrow$

$$u_{xx}(x, y) \approx \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2},$$

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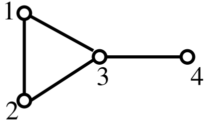

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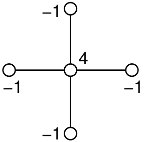

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- weighted Laplacian for $w \geq 0$: $L_w(G) = \sum_{ij \in E} w_{ij} E_{ij} \quad E_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$
- Properties: $L_w \succeq 0$ (sym., pos. semidef.), $\lambda_1(L_w) = 0$ with EV $\mathbf{1}$.

Nodal Domains

In the wave equation of a string or drum, eigenfunctions represent modes of vibration.

For each eigenfunction, the zeros represent nodes/nodal lines, along which the string/drum does not move. The moving regions in between are the nodal domains, all points in a nodal domain have the same sign.

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For an eigenvector v of $L_w(G)$ of a connected graph $G = (N, E)$ there are

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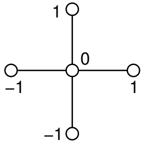

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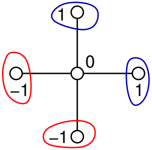

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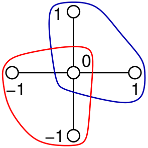

For the depicted values of the eigenvector there are 4 strong nodal domains and 2 weak nodal domains.

Nodal Domains of Generalized Laplacians [BLS2000]

A symmetric matrix M is called a generalized Laplacian or discrete Schrödinger operator of a graph $G = (N, E)$, if

$$M_{ij} \text{ is } \begin{cases} < 0 & \text{for } ij \in E, \\ = 0 & \text{for } ij \notin E, i \neq j, \\ \in \mathbb{R} & \text{for } i = j. \end{cases}$$

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Fiedler1974 proved that for any eigenvector v of $\lambda_2(G)$ and $\alpha \leq 0$, the node set $N_\alpha := \{i \in N : v_i \geq \alpha\}$ induces a connected subgraph. He showed tight relations between $\lambda_2(G)$ and the connectivity of the graph.
$\rightarrow$ Fiedler-vector in spectral graph partitioning heuristics.

Rough sketch for strong nodal domains: idea goes back to Courant.

Let v^k to λ_k induce strong domains $D_1,\ldots,D_m$.

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Now $\quad \lambda_m \leq g^T M g \leq \lambda_k < \lambda_{k+r} \qquad$ implies $m < k + r$.

The Colin de Verdière Graph Parameter $\mu(G)$ [CdV199*,vdHLS1999]

Given a graph $G = (N, E)$ the parameter $\mu(G)$ is the largest corank of any matrix M satisfying

(M1) M is a generalized Laplacian

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$\mu(G)$ is bounded above by the tree width $+1$ and it is minor monotone!

- $\mu(G) \leq 1 \Leftrightarrow G$ is a disjoint union of paths
- $\mu(G) \leq 2 \Leftrightarrow G$ is outerplanar
- $\mu(G) \leq 3 \Leftrightarrow G$ is planar
- $\mu(G) \leq 4 \Leftrightarrow G$ is linklessly embeddable

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$$\langle A_i, X\rangle = b_i, \qquad i = 1, \ldots, m.$$

For $r = \min\{\sqrt{2m}, n\}$ and any $d \geq 1$ there is an $X_0 \succeq 0$, $\mathrm{rank}(X_0) \leq d$ so that

$$\beta(m,n,d) \cdot b_i \leq \langle A_i, X_0\rangle \leq \alpha(m,n,d) \cdot b_i \qquad i = 1, \ldots, m$$

$$\text{with} \quad \alpha(m,n,d) = \begin{cases} 1 + \frac{12\ln(4mr)}{d} & \text{for } 1 \leq d \leq 12\ln(4mr) \\ 1 + \sqrt{\frac{12\ln(4mr)}{d}} & \text{for } d > 12\ln(4mr) \end{cases}$$

$$\text{and} \quad \beta(m,n,d) = \begin{cases} \frac{1}{(2m)^{2/d}e} & \text{for } 1 \leq d \leq 4\ln(2m) \\ \max\left\{\frac{1}{(2m)^{2/d}e}, 1 - \sqrt{\frac{4\ln(2m)}{d}}\right\} & \text{for } d > 4\ln(2m). \end{cases}$$

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$$\text{with} \quad \alpha(m,n,d) = \begin{cases} 1 + \frac{12\ln(4mr)}{d} & \text{for } 1 \leq d \leq 12\ln(4mr) \\ 1 + \sqrt{\frac{12\ln(4mr)}{d}} & \text{for } d > 12\ln(4mr) \end{cases}$$

$$\text{and} \quad \beta(m,n,d) = \begin{cases} \frac{1}{(2m)^{2/d}e} & \text{for } 1 \leq d \leq 4\ln(2m) \\ \max\left\{\frac{1}{(2m)^{2/d}e}, 1 - \sqrt{\frac{4\ln(2m)}{d}}\right\} & \text{for } d > 4\ln(2m). \end{cases}$$

SDP Rank Reduction [SoYeZhang2006]

- Given distance constraints $\langle E_{ij}, X\rangle = d_{ij}$, how much do we have to violate them to find low-rank solutions?

Theorem. Given $A_1, \ldots, A_m \in \mathcal{S}_+^n$, $b_1, \ldots, b_m \in \mathbb{R}_+$, suppose there is an $X \succeq 0$ with

$$\langle A_i, X\rangle = b_i, \qquad i = 1, \ldots, m.$$

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- [Barvinok2002]: $(1-\varepsilon)b_i \leq \langle A_i, X_0\rangle \leq (1+\varepsilon)b_i$ for $d = 8\varepsilon^{-2}\ln(4m)$
- includes [JohnsonLindenstrauss1984]

Finding Low-Rank Solutions by Norm Minimization [RechtFazelParrilo2007]

- rank version of sparsest vector/compressed sensing is

$$\min \ \operatorname{rank}(Y) \quad \text{s.t.} \quad \mathcal{A}Y = b, \ Y \in \mathbb{R}^{m \times n}$$

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How to represent $X \succeq 0$?

$$\mathrm{rank}\left(Y = \begin{bmatrix} I_d & V \\ V^T & X \end{bmatrix}\right) = d \qquad \Leftrightarrow \qquad X = V^TV \succeq 0$$

$\Rightarrow$ EDM-Completion in $\mathbb{R}^d$ may be cast as the problem

$$\min\ \mathrm{rank}(Y)\ \text{s.t.}\ \mathcal{A}Y = b$$

Nuclear Norm Minimization [RechtFazelParrilo2007]

- Solve relaxation of $\min \operatorname{rank}(Y)$ s.t. $\mathcal{A}Y = b, \ Y \in \mathbb{R}^{m \times n}$
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$\Rightarrow$ diagonal matrices and for each i $\begin{bmatrix} \alpha_i & \sigma_i \\ \sigma_i & \beta_i \end{bmatrix} \succeq 0 \quad \Rightarrow \quad \alpha_i\beta_i \geq \sigma_i^2$

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- SDPLR by [BurerMonteiro2003/5]